Matrice

Neka su m i n pozitivni cijeli brojevi.
m×n matrica je kolekcija od m·n brojeva uređenih u pravougaoni niz:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}}_{n \text{ kolona}} \; m \text{ redova}$$

Npr. $\begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & -5 \end{bmatrix}$ je 2×3 matrica, $A = \begin{bmatrix} 1 & \sqrt{2} & 8 & 9 \\ 7 & 2 & -5 & 3 \\ 4 & -6 & 7 & 8 \\ 3 & 7 & 2 & 8 \\ 1 & 2 & -2 & 5 \end{bmatrix}_{5\times 4}$

Brojeve u matrici zovemo elementi matrice i označavamo sa a_{ij}, gdje su i, j cijeli $1 \le i \le m$ i $1 \le j \le n$. Indeks i zovemo red indeks, a j kolona indeks.

$$\begin{array}{cc} & j \\ i & \begin{bmatrix} & \vdots & \\ \dots & a_{ij} & \dots\dots \\ & \vdots & \\ & \vdots & \end{bmatrix} \end{array}$$

Npr. u matrici A

$a_{12} = \sqrt{2}$, $a_{23} = -5$, $a_{43} = 2$, $a_{53} = -2$

1×n matricu zovemo n-dimenzionalni red vektor, $A = [a_1 \dots a_n]$
m×1 matrica je m-dimenzionalni kolona vektor $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

Sabiranje matrica: $[a_{ij}]_{m\times n} + [b_{ij}]_{m\times n} = [s_{ij}]_{m\times n}$,
gdje je $s_{ij} = a_{ij} + b_{ij}$, $\forall i, j$

npr.

$$\begin{bmatrix} 2 & 1 & 0 & 3 \\ 4 & 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 & 3 \\ 4 & 1 & 3 & 4 \end{bmatrix}$$

Skalarno množenje matrice brojem:

c je realan broj $\quad c\cdot[a_{ij}]_{m\times n} = [b_{ij}]_{m\times n}$

gdje je $b_{ij} = c\cdot a_{ij} \ \forall i,j$

npr.

$$2\begin{bmatrix}0 & 1\\ 2 & 3\\ 2 & 1\end{bmatrix} = \begin{bmatrix}0 & 2\\ 4 & 6\\ 4 & 2\end{bmatrix}$$

Brojeve ćemo često zvati skalari.

Množenje matrica:

Prvo ćemo vidjeti šta je proizvod red vektora A i kolone vektora B.

$$A\cdot B = [a_1\ a_2 \dots a_n]\begin{bmatrix}b_1\\ b_2\\ \vdots\\ b_n\end{bmatrix} = a_1b_1 + a_2b_2 + \dots + a_nb_n.$$

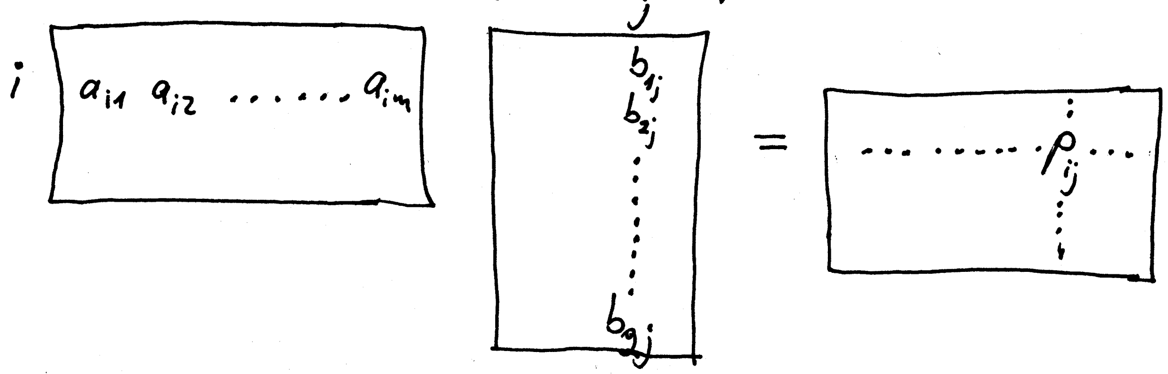
npr. $[3\ \ 1\ \ 2]\begin{bmatrix}1\\ -1\\ 4\end{bmatrix} = 3-1+8=10$

generalno:

$$[a_{ij}]_{m\times q}\cdot[b_{ij}]_{q\times s} = [p_{ij}]_{m\times s} \quad \text{gdje je}$$

$$p_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{im}b_{mj}$$

ovo znači proizvod i-tog reda od A i j-te kolone od B.



npr. $\begin{bmatrix}0 & -1 & 2\\ 3 & 4 & -6\end{bmatrix}\begin{bmatrix}1\\ 4\\ 3\end{bmatrix} = \begin{bmatrix}2\\ 1\end{bmatrix}$

Sistem linearnih jednačina

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1\\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2\\ \vdots \qquad \vdots \qquad\quad \vdots \qquad &\ \ \vdots\\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

možemo pisati u matričnom obliku $Ax=\mathbf{b}$, gdje A predstavlja koeficijent matricu $[a_{ij}]_{m\times n}$

$$\boxed{\quad A \quad}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} = \begin{bmatrix}b_1\\ \vdots\\ b_m\end{bmatrix}.$$

1. Ako je $A=\begin{bmatrix}2&4&5\\3&2&6\\1&1&7\end{bmatrix}$ i $B=\begin{bmatrix}1&-1&6\\3&0&4\\5&2&10\end{bmatrix}$ izračunati:

a) $A+B$ b) $A-B$ c) $2A-3B-I$ (I jedinična matrica)

Rj. a) $\begin{bmatrix}2&4&5\\3&2&6\\1&1&7\end{bmatrix}+\begin{bmatrix}1&-1&6\\3&0&4\\5&2&10\end{bmatrix}=\begin{bmatrix}3&3&11\\6&2&10\\6&3&17\end{bmatrix}$

b) $\begin{bmatrix}2&4&5\\3&2&6\\1&1&7\end{bmatrix}-\begin{bmatrix}1&-1&6\\3&0&4\\5&2&10\end{bmatrix}=\begin{bmatrix}1&5&-1\\0&2&2\\-4&-1&-3\end{bmatrix}$

c) $2\begin{bmatrix}2&4&5\\3&2&6\\1&1&7\end{bmatrix}-3\begin{bmatrix}1&-1&6\\3&0&4\\5&2&10\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}4&8&10\\6&4&12\\2&2&14\end{bmatrix}-\begin{bmatrix}3&-3&18\\9&0&12\\15&6&30\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&11&-8\\-3&3&0\\-13&-4&-17\end{bmatrix}$

2. Izračunati:

a) $\begin{bmatrix}2&3\\1&6\\0&1\end{bmatrix}\cdot\begin{bmatrix}2&1\\3&5\end{bmatrix}$ Rj. $=\begin{bmatrix}2\cdot2+3\cdot3&2\cdot1+3\cdot5\\1\cdot2+6\cdot3&1\cdot1+6\cdot5\\0\cdot2+1\cdot3&0\cdot1+1\cdot5\end{bmatrix}=\begin{bmatrix}13&17\\20&31\\3&5\end{bmatrix}$

b) $\begin{bmatrix}1&4\\2&-5\\3&6\end{bmatrix}\begin{bmatrix}1&4&-2\\2&5&6\end{bmatrix}=\begin{bmatrix}1\cdot1+4\cdot2&1\cdot4+4\cdot5&1\cdot(-2)+4\cdot6\\2\cdot1+(-5)\cdot2&2\cdot4+(-5)\cdot5&2\cdot(-2)+(-5)6\\3\cdot1+6\cdot2&3\cdot4+6\cdot5&3\cdot(-2)+6\cdot6\end{bmatrix}=\begin{bmatrix}9&24&22\\-8&-17&-34\\15&42&30\end{bmatrix}$

c) $\begin{bmatrix}2\\1\\3\end{bmatrix}\begin{bmatrix}1&2&3\end{bmatrix}=\begin{bmatrix}2&4&6\\1&2&3\\3&6&9\end{bmatrix}$

d) $\begin{bmatrix}a&b&c\end{bmatrix}\cdot\begin{bmatrix}1\\2\\3\end{bmatrix}=a+2b+3c$

3. Ako je $A=\begin{bmatrix}1&-2&3\\2&-4&1\\3&-5&2\end{bmatrix}$ izračunati $3A^2-2A^T+5\cdot I$.

(A^T transponovana matrica matrice A) (kada elementi iz reda zamjene položaj sa elementima iz kolona)

Rj. $A^T=\begin{bmatrix}1&2&3\\-2&-4&-5\\3&1&2\end{bmatrix}$. $A^2=\begin{bmatrix}1&-2&3\\2&-4&1\\3&-5&2\end{bmatrix}\begin{bmatrix}1&-2&3\\2&-4&1\\3&-5&2\end{bmatrix}=\begin{bmatrix}6&-9&7\\-3&7&4\\-1&4&8\end{bmatrix}$

$3A^2-2A^T+5=\begin{bmatrix}18&-27&21\\-9&21&12\\-3&12&24\end{bmatrix}-\begin{bmatrix}2&4&6\\-4&-8&-10\\6&2&4\end{bmatrix}+\begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix}=\begin{bmatrix}21&-31&15\\-5&34&22\\-9&10&25\end{bmatrix}$

4. Ako je $A=\begin{bmatrix}2&3&5\\-3&1&5\end{bmatrix}$ i $B=\begin{bmatrix}-2&-3\\-1&0\\1&1\end{bmatrix}$, izračunati $2\cdot A^T\cdot A-3\cdot B\cdot B^T+6I$.

Rj. $\begin{bmatrix}-7&0&5\\0&23&43\\5&43&100\end{bmatrix}$

Determinante

matrica tipa $n\times n$

Determinanta je broj pridružen svakoj kvadratnoj matrici.
Determinantu matrice A obilježavamo sa $\det A$ ili $|A|$.
Preciznija definicija determinante je:
Determinanta je f-ja koja $n\times n$ realnih brojeva preslikava u realan broj.
Osobine determinante: (neke osobine determinanti)

1. Determinanta jedinične matrice je 1 ($\det I = 1$).
2. Ako dva reda (ili dvije kolone) međusobno zamjene mjesta znak determinante se mijenja.

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \quad \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1, \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

3. a) Determinanta se množi jednim brojem ako se tim brojem pomnože svi elementi jednog reda (ili jedne kolone)

$$t\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} ta & tb \\ c & d \end{vmatrix}$$

b) $$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

(linearnost za svaki red)

1. Izračunati:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

a) $$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 0 \end{vmatrix} \overset{R_3}{=} 2\cdot\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - 0\cdot\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 0\cdot\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 2\cdot 1 = 2$$

razvoj determinante po trećem redu

b) $$\begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 3 & 1 & 1 \end{vmatrix} \xlongequal{\text{razvoj determinante po prvom redu}} 1\cdot\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 2\cdot\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} + 0\cdot\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix}$$

$$= 1\cdot 0 - 2\cdot(-3) + 0 = 6$$

Mogli smo izračunati i na sljedeći način:

$$\begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 3 & 1 & 1 \end{vmatrix} \xlongequal{II_k - III_k} \begin{vmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 3 & 0 & 1 \end{vmatrix} = (-2)\cdot\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = (-2)\cdot(-3) = 6$$

2. Izračunati:

a) $\begin{vmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix} \xlongequal{III_k - IV_k} \begin{vmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix} = (-1) \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = -2$

b) $\begin{vmatrix} 4 & 1 & 0 & 3 \\ 2 & 0 & 1 & 1 \\ 4 & 1 & 0 & 1 \\ 0 & 1 & 0 & 3 \end{vmatrix} \xlongequal{I_R - IV_R} \begin{vmatrix} 4 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 4 & 1 & 0 & 1 \\ 0 & 1 & 0 & 3 \end{vmatrix} = 4 \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 3 \end{vmatrix} = 4 \cdot (-1) \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = (-4) \cdot 2 =$

$= -8$

3. Izračunati:

a) $\begin{vmatrix} 3 & -2 & 1 \\ 4 & -1 & 1 \\ 1 & 1 & 5 \end{vmatrix}$ Rj. $\xlongequal{I_R - II_R} \begin{vmatrix} -1 & -1 & 0 \\ 4 & -1 & 1 \\ 1 & 1 & 5 \end{vmatrix} \xlongequal{III_R + I_R} \begin{vmatrix} -1 & -1 & 0 \\ 4 & -1 & 1 \\ 0 & 0 & 5 \end{vmatrix} = 5 \begin{vmatrix} -1 & -1 \\ 4 & -1 \end{vmatrix}$

$= 5 \cdot 5 = 25$

b) $\begin{vmatrix} 1 & 3 & 3 \\ 2 & -1 & 4 \\ 1 & 2 & 7 \end{vmatrix} \xlongequal{I_R - III_R} \begin{vmatrix} 0 & 1 & -4 \\ 2 & -1 & 4 \\ 1 & 2 & 7 \end{vmatrix} \xlongequal{II_R + I_R} \begin{vmatrix} 0 & -1 & -4 \\ 2 & 0 & 0 \\ 1 & 2 & 7 \end{vmatrix} = (-2) \cdot \begin{vmatrix} 1 & -4 \\ 2 & 7 \end{vmatrix}$

$= (-2) \cdot 15 = -30$

4. Izračunati:

a) $\begin{vmatrix} 1 & 0 & 1 & 0 \\ 2 & 5 & 2 & 0 \\ 3 & 0 & 0 & 1 \\ 4 & 3 & 2 & 1 \end{vmatrix}$ Rj. $\xlongequal[\substack{III_R - I_R \cdot 3 \\ IV_R - I_R \cdot 4}]{II_R - I_R \cdot 2} \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 3 & -2 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 5 & 0 & 0 \\ 0 & -3 & 1 \\ 3 & -2 & 1 \end{vmatrix} = 5 \cdot \begin{vmatrix} -3 & 1 \\ -2 & 1 \end{vmatrix}$

$= 5 \cdot (-1) = -5$

b)^{v} $\begin{vmatrix} 0 & 0 & 1 & 2 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 5 \end{vmatrix}$ c)^{v} $\begin{vmatrix} 5 & 4 & 3 & 2 \\ 1 & 1 & 2 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \end{vmatrix}$

Rješenja:
b) 0 c) -1

5.^{v} Izračunati: $\begin{vmatrix} \sqrt{3} & 2\sqrt{2} & \sqrt{5} \\ 5\sqrt{3} & \sqrt{8} & 7\sqrt{5} \\ \sqrt{5}+2\sqrt{3} & 4\sqrt{2} & \sqrt{3}+2\sqrt{5} \end{vmatrix}$.

Rj. $36\sqrt{2}$

6. Dokazati da je $\begin{vmatrix} 1 & a & a^2+a^3 \\ 1 & a^2 & a^3+a \\ 1 & a^3 & a+a^2 \end{vmatrix} = 0$.

Rj.

$$\begin{vmatrix} 1 & a & a^2+a^3 \\ 1 & a^2 & a^3+a \\ 1 & a^3 & a+a^2 \end{vmatrix} = a\begin{vmatrix} 1 & 1 & a^2(1+a) \\ 1 & a & a(a^2+1) \\ 1 & a^2 & a(1+a) \end{vmatrix} = a\cdot a\cdot\begin{vmatrix} 1 & 1 & a(a+1) \\ 1 & a & a^2+1 \\ 1 & a^2 & a+1 \end{vmatrix} \underset{III_R - I_R}{\overset{II_R - I_R}{=}}$$

$$= a^2\begin{vmatrix} 1 & 1 & a(a+1) \\ 0 & a-1 & 1-a \\ 0 & a^2-1 & 1-a^2 \end{vmatrix} = a^2\begin{vmatrix} a-1 & 1-a \\ (a+1)(a-1) & 1-a^2 \end{vmatrix} = a^2(a-1)\begin{vmatrix} 1 & 1-a \\ a+1 & (1-a)(1+a) \end{vmatrix}$$

$$= a^2(a-1)(1-a)\begin{vmatrix} 1 & 1 \\ a+1 & a+1 \end{vmatrix} = a^2(a-1)(1-a)(a+1)\underbrace{\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}}_{=0} = 0$$

što je i trebalo dobiti.

7. Izračunati: $\begin{vmatrix} a & b & a & b \\ b & a & a & b \\ a & b & b & a \\ b & a & b & a \end{vmatrix}$

Rj. $\overset{IV_k+(I_k+II_k+III_k)}{=} \begin{vmatrix} a & b & a & 2a+2b \\ b & a & a & 2a+2b \\ a & b & b & 2a+2b \\ b & a & b & 2a+2b \end{vmatrix}$

$$=(2a+2b)\begin{vmatrix} a & b & a & 1 \\ b & a & a & 1 \\ a & b & b & 1 \\ b & a & b & 1 \end{vmatrix} \underset{IV_R - III_R}{\overset{II_R - I_R}{=}} (2a+2b)\begin{vmatrix} a & b & a & 1 \\ b-a & a-b & 0 & 0 \\ a & b & b & 1 \\ b-a & a-b & 0 & 0 \end{vmatrix} \overset{III_R - I_R}{=} (2a+2b)\cdot$$

$$\cdot\begin{vmatrix} a & b & a & 1 \\ b-a & a-b & 0 & 0 \\ 0 & 0 & b-a & 1 \\ b-a & a-b & 0 & 0 \end{vmatrix} = -(2a+2b)\cdot\begin{vmatrix} a & b & a \\ b-a & a-b & 0 \\ b-a & a-b & 0 \end{vmatrix} = -a(2a+2b)\begin{vmatrix} b-a & a-b \\ b-a & a-b \end{vmatrix}$$

$$= -a(2a+2b)(b-a)(a-b)\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0.$$

8. Rastaviti na faktore polinom:

a) $\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$ b) $\begin{vmatrix} a & b & a+b \\ b & a+b & a \\ a+b & a & b \end{vmatrix}$ c) $\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 \\ b^2 & (b+1)^2 & (b+2)^2 \\ c^2 & (c+1)^2 & (c+2)^2 \end{vmatrix}$.

\# Riješiti jednačinu $\begin{vmatrix} 3x-5 & -5-2x & x+1 \\ 2x-4 & -2-2x & x-1 \\ 3x-8 & 2-3x & 2x-5 \end{vmatrix} = 0$.

Rj.

$$\begin{vmatrix} 3x-5 & -5-2x & x+1 \\ 2x-4 & -2-2x & x-1 \\ 3x-8 & 2-3x & 2x-5 \end{vmatrix} = (-1)\begin{vmatrix} 3x-5 & 2x+5 & x+1 \\ 2x-4 & 2x+2 & x-1 \\ 3x-8 & 3x-2 & 2x-5 \end{vmatrix} \xlongequal{III_v - II_v}$$

$$\begin{vmatrix} 3x-5 & 2x+5 & x+1 \\ 2x-4 & 2x+2 & x-1 \\ x-4 & x-4 & x-4 \end{vmatrix} = (-1)(x-4)\begin{vmatrix} 3x-5 & 2x+5 & x+1 \\ 2x-4 & 2x+2 & x-1 \\ 1 & 1 & 1 \end{vmatrix} \xlongequal[II_k - III_k]{I_k - III_k}$$

$$= (-1)(x-4)\begin{vmatrix} 2x-6 & x+4 & x+1 \\ x-3 & x+3 & x-1 \\ 0 & 0 & 1 \end{vmatrix} = (-1)(x-4)\begin{vmatrix} 2x-6 & x+4 \\ x-3 & x+3 \end{vmatrix} \xlongequal{I_v - II_v}$$

$$= (-1)(x-4)\begin{vmatrix} x-3 & 1 \\ x-3 & x+3 \end{vmatrix} = (-1)(x-4)(x-3)\begin{vmatrix} 1 & 1 \\ 1 & x+3 \end{vmatrix} = (-1)(x-4)(x-3)(x+2)$$

$(-1)(x-4)(x-3)(x+2) = 0$ Rješenja jednačine su $x=4$ ili $x=3$ ili $x=-2$.

\# Riješiti jednačinu: $\begin{vmatrix} x-3 & x+2 & x-1 \\ x+2 & x-4 & x \\ x-1 & x+4 & x-5 \end{vmatrix} = 0$.

Rj.

$$\begin{vmatrix} x-3 & x+2 & x-1 \\ x+2 & x-4 & x \\ x-1 & x+4 & x-5 \end{vmatrix} \xlongequal{I_R + II_R + III_R} \begin{vmatrix} 3x-2 & x+2 & x-1 \\ 3x-2 & x-4 & x \\ 3x-2 & x+4 & x-5 \end{vmatrix} = (3x-2)\begin{vmatrix} 1 & x+2 & x-1 \\ 1 & x-4 & x \\ 1 & x+4 & x-5 \end{vmatrix}$$

$$\xlongequal[III_R - II_R]{I_R - II_R} (3x-2)\begin{vmatrix} 0 & 6 & -1 \\ 1 & x-4 & x \\ 0 & 8 & -5 \end{vmatrix} = -(3x-2)\begin{vmatrix} 6 & -1 \\ 8 & -5 \end{vmatrix} = -(3x-2)(-30+8) =$$

$= 22(3x-2)$ $\quad 22(3x-2) = 0$ $\quad 3x-2=0$ $\quad 3x = 2$ $\quad x = \frac{2}{3}$ je rješenje jednačine

#) Izračunati $\begin{vmatrix} 1 & a & 3 & 2 \\ 2 & 2 & -2 & 1 \\ 3 & 3 & -5 & 1 \\ 4 & 4 & -7 & 5 \end{vmatrix}$.

Rj.

$$\begin{vmatrix} 1 & a & 3 & 2 \\ 2 & 2 & -2 & 1 \\ 3 & 3 & -5 & 1 \\ 4 & 4 & -7 & 5 \end{vmatrix} \underset{III_k + IV_k \cdot 2}{\overset{\substack{I_k + III_k \\ II_k + III_k}}{=\!=\!=}} \begin{vmatrix} 4 & a+3 & 7 & 2 \\ 0 & 0 & 0 & \boxed{1} \\ -2 & -2 & -3 & 1 \\ -3 & -3 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 4 & a+3 & 7 \\ -2 & -2 & -3 \\ -3 & -3 & 3 \end{vmatrix} \underset{II_k + III_k}{\overset{I_k + III_k}{=\!=\!=}}$$

$$= \begin{vmatrix} 11 & a+10 & 7 \\ -5 & -5 & -3 \\ 0 & 0 & 3 \end{vmatrix} = 3 \begin{vmatrix} 11 & a+10 \\ -5 & -5 \end{vmatrix} = 3 \cdot (-5) \begin{vmatrix} 11 & a+10 \\ 1 & 1 \end{vmatrix} = -15\,(11 - a - 10)$$

$$= -15\,(-a+1) = 15a - 15$$

(#) Matematičkom indukcijom dokazati:

$$\begin{vmatrix} 1+x^2 & x & 0 & \dots & 0 & 0 \\ x & 1+x^2 & x & \dots & 0 & 0 \\ 0 & x & 1+x^2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1+x^2 & x \\ 0 & 0 & 0 & \dots & x & 1+x^2 \end{vmatrix} = 1+x^2+x^4+\dots+x^{2n}$$

(determinanta ima n vrsta i n kolona).

R: BAZA INDUKCIJE

Pokažimo da je tvrdnja tačna za broj 2

$$\begin{vmatrix} 1+x^2 & x \\ x & 1+x^2 \end{vmatrix} = (1+x^2)^2 - x^2 = 1+2x^2+x^4-x^2 = 1+x^2+x^4$$

Jednakost je tačna za broj 2.

KORAK INDUKCIJE

Pretpostavimo da je jednakost tačna za determinantu koja ima k vrsta i k kolona

$$\begin{vmatrix} 1+x^2 & x & \dots & 0 & 0 \\ x & 1+x^2 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1+x^2 & x \\ 0 & 0 & \dots & x & 1+x^2 \end{vmatrix} = 1+x^2+x^4+\dots+x^{2k}$$

gdje k uzima brojeve od 1 do n

Na osnovu ove pretpostavke dokažimo da je jednakost tačna za determinantu koja ima $n+1$ vrstu i $n+1$ kolonu tačnije dokažimo da

$$\begin{vmatrix} 1+x^2 & x & \dots & 0 & 0 \\ x & 1+x^2 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1+x^2 & x \\ 0 & 0 & \dots & x & 1+x^2 \end{vmatrix} = 1+x^2+x^4+\dots+x^{2n}+x^{2n+2}.$$

Polazimo od determinante koja ima $(n+1)$-vrstu i $(n+1)$-kolonu:

$$\begin{vmatrix} 1+x^2 & x & \dots & 0 & 0 \\ x & 1+x^2 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1+x^2 & x \\ 0 & 0 & \dots & x & 1+x^2 \end{vmatrix} \xlongequal[\text{koloni}]{\text{razvoj po prvoj}} (1+x^2) \begin{vmatrix} 1+x^2 & x & \dots & 0 & 0 \\ x & 1+x^2 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1+x^2 & x \\ 0 & 0 & \dots & x & 1+x^2 \end{vmatrix} - x \begin{vmatrix} x & 0 & 0 & \dots & 0 & 0 \\ x & 1+x^2 & x & \dots & 0 & 0 \\ 0 & x & 1+x^2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1+x^2 & x \\ 0 & 0 & 0 & \dots & x & 1+x^2 \end{vmatrix} =$$

(ova determinanta ima n vrsta i n kolona)

(ovu determinantu mogu razviti po prvoj vrsti i ostaće mi determinanta iz pretpostavke koja ima $n-1$ vrstu i $n-1$ kolonu)

$$\xlongequal{\text{na osnovu pretpostavke}} (1+x^2)(1+x^2+x^4+\dots+x^{2n}) - x^2(1+x^2+x^4+\dots+x^{2n-2})$$

$$= (1+x^2+x^4+\dots+x^{2n}) + (x^2+x^4+x^6+\dots+x^{2n}+x^{2n+2}) -$$

$$-(x^2+x^4+x^6+\dots+x^{2n-2}+x^{2n}) = 1+x^2+x^4+\dots+x^{2n+2}$$ što je i trebalo dobiti

ZAKLJUČAK

Jednakost je tačna za sve prirodne brojeve

(#) Matematičkom indukcijom dokazati:

$$\begin{vmatrix} 1 & n & n & \dots & n & n \\ n & 2 & n & \dots & n & n \\ n & n & 3 & \dots & n & n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n & n & n & \dots & n-1 & n \\ n & n & n & \dots & n & n \end{vmatrix} = (-1)^{n-1}\cdot n!$$

Rj. BAZA INDUKCIJE

Pokažimo da je tvrdnja tačna za broj 2.

$$\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = 2-4 = -2 = (-1)^{2-1}\cdot 2!$$ Jednakost je tačna za broj 2.

KORAK INDUKCIJE

Pretpostavimo da je jednakost

$$\begin{vmatrix} 1 & k & k & \dots & k & k \\ k & 2 & k & \dots & k & k \\ k & k & 3 & \dots & k & k \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ k & k & k & \dots & k-1 & k \\ k & k & k & \dots & k & k \end{vmatrix} = (-1)^{k-1}\cdot k!$$

tačna za sve brojeve od 1 do n $(k=1,2,\dots,n)$.

Uz pomoć ove pretpostavke dokažimo da je jednakost tačna za broj $n+1$ tj. dokažimo

$$\begin{vmatrix} 1 & n+1 & \dots & n+1 & n+1 \\ n+1 & 2 & \dots & n+1 & n+1 \\ \vdots & \vdots & & \vdots & \vdots \\ n+1 & n+1 & \dots & n & n+1 \\ n+1 & n+1 & \dots & n+1 & n+1 \end{vmatrix} = (-1)^{n}\cdot(n+1)!$$

$$\boxed{\begin{array}{l} \text{ZAKLJUČAK} \\ \text{Jednakost je} \\ \text{tačna za sve} \\ \text{prirodne brojeve} \end{array}}$$

$$\begin{vmatrix} 1 & n+1 & \dots & n+1 & n+1 \\ n+1 & 2 & \dots & n+1 & n+1 \\ \vdots & \vdots & & \vdots & \vdots \\ n+1 & n+1 & \dots & n & n+1 \\ n+1 & n+1 & \dots & n+1 & n+1 \end{vmatrix} \xrightarrow{I_k-(N+1)_k} \begin{vmatrix} -n & n+1 & \dots & n+1 & n+1 \\ 0 & 2 & \dots & n+1 & n+1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & n+1 & \dots & n & n+1 \\ 0 & n+1 & \dots & n+1 & n+1 \end{vmatrix} =$$

$$=(-n)\begin{vmatrix} 2 & n+1 & \dots & n+1 & n+1 \\ n+1 & 3 & \dots & n+1 & n+1 \\ \vdots & \vdots & & \vdots & \vdots \\ n+1 & n+1 & \dots & n & n+1 \\ n+1 & n+1 & \dots & n+1 & n+1 \end{vmatrix} = (-n)(n+1)\begin{vmatrix} 2 & n+1 & \dots & n+1 & 1 \\ n+1 & 3 & \dots & n+1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ n+1 & n+1 & \dots & n & 1 \\ n+1 & n+1 & \dots & n+1 & 1 \end{vmatrix} \xrightarrow[\substack{\vdots \\ (N-1)_k - N_k}]{\substack{I_k - N_k \\ II_k - N_k}}$$

$$=(-1)\cdot n(n+1)\begin{vmatrix} 1 & n & \dots & n & 1 \\ n & 2 & \dots & n & 1 \\ \vdots & \vdots & & & \vdots \\ n & n & \dots & n-1 & 1 \\ n & n & \dots & n & 1 \end{vmatrix} = (-1)(n+1)\begin{vmatrix} 1 & n & \dots & n & n \\ n & 2 & \dots & n & n \\ \vdots & \vdots & & \vdots & \vdots \\ n & n & \dots & n-1 & n \\ n & n & \dots & n & n \end{vmatrix} \underset{\text{pretpostavke}}{\overset{\text{na osnovu}}{=}} (-1)(n+1)(-1)^{n-1}\cdot n! = (-1)^n(n+1)!$$